10 Pasch Geometries

Definition (Pasch's Postulate (PP))
A metric geometry satisfies Pasch's Postulate (PP) if for any line ℓ, any triangle $\triangle ABC$, and any point $D \in \ell$ such that $A - D - B$, then either $\ell \cap \overline{AC} \neq \emptyset$ or $\ell \cap \overline{BC} \neq \emptyset$.

Theorem (Pasch's Theorem) If a metric geometry satisfies PSA then it also satisfies PP.

1. Prove the above theorem.

Definition (Pasch Geometry)
A Pasch Geometry is a metric geometry which satisfies PSA.

Theorem Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry which satisfies PP. If A, B, C are noncollinear and if the line ℓ does not contain any of the points A, B, C, then ℓ cannot intersect all three sides of $\triangle ABC$.

2. Prove the above theorem.

Theorem If a metric geometry satisfies PP then it also satisfies PSA.

3. Prove the above theorem.

4. (Peano's Axiom) Given a triangle $\triangle ABC$ in a metric geometry which satisfies PSA and points D, E with $B - C - D$ and $A - E - C$, prove there is a point $F \in \overleftrightarrow{DE}$ with $A - F - B$, and $D - E - F$.

5. Given $\triangle ABC$ in a metric geometry which satisfies PSA and points D, F with $B - C - D$, $A - F - B$, prove there exists $E \in \overleftrightarrow{DF}$ with $A - E - C$ and $D - E - F$.

6. Given $\triangle ABC$ and a point P in a metric geometry which satisfies PSA prove there is a line through P that contains exactly two points of $\triangle ABC$.

Definition (Missing Strip Plane)
The Missing Strip Plane is the abstract geometry $\{\mathcal{S}, \mathcal{L}\}$ given by
$\mathcal{S} = \{(x, y) \in \mathbb{R}^2 \mid x < 0 \text{ or } 1 \leq x\}$,
$\mathcal{L} = \{\ell \cap \mathcal{S} \mid \ell \text{ is a Cartesian line and } \ell \cap \mathcal{S} \neq \emptyset\}$.

7. Given the following pairs of points: (i) $(2, 3)$ and $(3, -1)$; (ii) $(0, 3)$ and $(1/2, -2)$; (iii) $(-1, 4)$ and $(2, 7)$. If the given pair of points lies in the point set of the Missing Strip Plane, find the line through that pair of points.

8. If lines ℓ_1, ℓ_2 and ℓ_3 in the Missing Strip plane satisfy:
ℓ_1 is parallel to ℓ_2 and
ℓ_2 is parallel to ℓ_3,
is it true that ℓ_1 is parallel to ℓ_3? Justify your answer.

9. Given that a metric geometry satisfies PSA if and only if it is a Pasch geometry, give an example to show that the Missing Strip Plane does not satisfy PSA.

10. Let $\mathcal{S}$ denote the set of points of the Missing Strip plane. Find all lines in this plane through the point $(2, 0)$ which are parallel in the Missing Strip plane to (i) the line $L_{-1} \cap \mathcal{S}$; (ii) the line $L_{1,2} \cap \mathcal{S}$.

11. Prove that the Missing Strip Plane is an incidence geometry.

Proposition If $\{\mathcal{S}, \mathcal{L}\}$ is the Missing Strip Plane and $\ell = L_{m,b}$ then $g_\ell : \ell \cap \mathcal{S} \to \mathbb{R}$ is a bijection (for definition of g_ℓ see lecture notes or in book on page 79).

12. Prove the above proposition.

Proposition The Missing Strip Plane is not a Pasch geometry.

13. Prove the above proposition.

14. Let $\mathcal{S}$ denote the set of points of the Missing Strip plane. Find all lines in this plane through the point $(-1, 1)$ which are parallel in the Missing Strip plane to (i) the line $L_2 \cap \mathcal{S}$; (ii) the line $L_{-1,2} \cap \mathcal{S}$.

15. Given a triangle, $\triangle ABC$, in a metric geometry, and points D, E with $A - D - B$ and $C - E - B$, is it always the case that $\overleftrightarrow{AE} \cap \overleftrightarrow{CD} \neq \emptyset$?

11 Interiors and the Crossbar Theorem

Theorem In a Pasch geometry if $\mathcal{A}$ is a non-empty convex set that does not intersect the line ℓ, then all points of $\mathcal{A}$ lie on the same side of ℓ.

1. Prove the above theorem.

> **Definition (interior of the ray, interior of the segment)**
> The interior of the ray $\overrightarrow{AB}$ in a metric geometry is the set $\text{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\}$. The interior of the segment $\overline{AB}$ in a metric geometry is the set $\text{int}(\overline{AB}) = \overline{AB} - \{A, B\}$.

2. Prove that in a metric geometry, $\text{int}(\overrightarrow{AB})$ and $\text{int}(\overline{AB})$ are convex sets.

Theorem Let $\mathcal{A}$ be a line, ray, segment, the interior of a ray, or the interior of a segment in a Pasch geometry. If ℓ is a line with $\mathcal{A} \cap \ell = \emptyset$ then all of $\mathcal{A}$ lies on one side of ℓ. If there is a point B with $A - B - C$ and $\overleftrightarrow{AC} \cap \ell = \{B\}$ then $\text{int}(\overrightarrow{BA})$ and $\text{int}(\overline{BA})$ both lie on the same side of ℓ while $\text{int}(\overrightarrow{BA})$ and $\text{int}(\overrightarrow{BC})$ lie on opposite sides of ℓ.

3. Prove the above theorem.

Theorem (Z Theorem) In a Pasch geometry, if P and Q are on opposite sides of the line $\overleftrightarrow{AB}$ then $\overrightarrow{BP} \cap \overrightarrow{AQ} = \emptyset$. In particular, $\overline{BP} \cap \overline{AQ} = \emptyset$.

4. Prove the above theorem.

> **Definition (interior of $\measuredangle ABC$)**
> In a Pasch geometry the interior of $\measuredangle ABC$, written $\text{int}(\measuredangle ABC)$, is the intersection of the side of $\overleftrightarrow{AB}$ that contains C with the side of $\overleftrightarrow{BC}$ that contains A.

Theorem In a Pasch geometry, if $\measuredangle ABC = \measuredangle A'B'C'$ then $\text{int}(\measuredangle ABC) = \text{int}(\measuredangle A'B'C')$.

5. Prove the above theorem.

Theorem In a Pasch geometry, $P \in \text{int}(\measuredangle ABC)$ if and only if A and P are on the same side of $\overleftrightarrow{BC}$ and C and P are on the same side of $\overleftrightarrow{BA}$.

6. Prove the above theorem.

Theorem Given $\triangle ABC$ in a Pasch geometry, if $A - P - C$ then $P \in \text{int}(\measuredangle ABC)$ and therefore $\text{int}(\overline{AC}) \subseteq \text{int}(\measuredangle ABC)$.

7. Prove the above theorem.

8. In a Pasch geometry, if $P \in \text{int}(\measuredangle ABC)$ prove $\text{int}(\overrightarrow{BP}) \subseteq \text{int}(\measuredangle ABC)$.

Theorem (Crossbar Theorem) In a Pasch geometry if $P \in \text{int}(\measuredangle ABC)$ then $\overrightarrow{BP}$ intersects $\overline{AC}$ at a unique point F with $A - F - C$.

9. Prove the above theorem.

Theorem In a Pasch geometry, if $\overline{CP} \cap \overleftrightarrow{AB} = \emptyset$ then $P \in \text{int}(\measuredangle ABC)$ if and only if A and C are on opposite sides of $\overleftrightarrow{BP}$.

10. Prove the above theorem.

Theorem In a Pasch geometry, if $A - B - D$ then $P \in \text{int}(\measuredangle ABC)$ if and only if $C \in \text{int}(\measuredangle DBP)$.

11. Prove the above theorem.

> **Definition (interior of $\triangle ABC$)**
> In a Pasch geometry, the interior of $\triangle ABC$, written $\text{int}(\triangle ABC)$, is the intersection of the side of $\overleftrightarrow{AB}$ which contains C, the side of $\overleftrightarrow{BC}$ which contains A, and the side of $\overleftrightarrow{CA}$ which contains B.

Theorem In a Pasch geometry $\text{int}(\triangle ABC)$ is convex.

12. Prove the above theorem.

13. In a Pasch geometry, given $\triangle ABC$ and points D, E, F such that $B - C - D$, $A - E - C$ and $B - E - F$, prove that $F \in \text{int}(\measuredangle ACD)$.

14. In a Pasch geometry, if $\overline{CP} \cap \overleftrightarrow{AB} = \emptyset$, prove that either $\overrightarrow{BC} = \overrightarrow{BP}$, or $P \in \text{int}(\measuredangle ABC)$, or $C \in \text{int}(\measuredangle ABP)$.

15. Prove that in a Pasch geometry, $\text{int}(\measuredangle ABC)$ is convex.